POINTWISE BEST APPROXIMATION RESULTS FOR GALERKIN FINITE ELEMENT SOLUTIONS OF PARABOLIC PROBLEMS

DMITRIY LEYKEKHMAN[†] AND BORIS VEXLER[‡]

Abstract. In this paper we establish a best approximation property of fully discrete Galerkin finite element solutions of second order parabolic problems on convex polygonal and polyhedral domains in the L^∞ norm. The discretization method uses of continuous Lagrange finite elements in space and discontinuous Galerkin methods in time of an arbitrary order. The method of proof differs from the established fully discrete error estimate techniques and for the first time allows to obtain such results in three space dimensions. It uses elliptic results, discrete resolvent estimates in weighted norms, and the discrete maximal parabolic regularity for discontinuous Galerkin methods established by the authors in [16]. In addition, the proof does not require any relationship between spatial mesh sizes and time steps. We also establish an interior best approximation property that shows a more local behavior of the error at a given point.

Key words. parabolic problems, finite elements, discontinuous Galerkin, a priori error estimates, pointwise error estimates

AMS subject classifications.

1. Introduction. Let Ω be a convex polygonal/polyhedral domains in $\mathbb{R}^N$, $N = 2, 3$ and $I = (0, T)$. We consider the second order parabolic problem

$$\begin{aligned} \partial_t u(t,x) - \Delta u(t,x) &= f(t,x), && (t,x) \in I \times \Omega, \\ u(t,x) &= 0, && (t,x) \in I \times \partial\Omega, \\ u(0,x) &= u_0(x), && x \in \Omega. \end{aligned} \tag{1.1}$$

For the purpose of this paper we assume that f and u_0 are such that the unique solution u of (1.1) fulfills $u \in C(\bar{I} \times \bar{\Omega}) \cap C(\bar{I}; H_0^1(\Omega))$. To achieve this, we can for example assume that the right-hand side $f \in L^r(I \times \Omega)$ with $r > \frac{N}{2} + 1$ and $u_0 \in C(\bar{\Omega}) \cap H_0^1(\Omega)$, cf., e. g., [42, Lemma 7.12], but other assumptions are possible.

To discretize the problem we use continuous Lagrange finite elements in space and discontinuous Galerkin methods in time. The precise description of the method is given in Section 2. Our main goal in this paper is to establish global and interior space-time pointwise best approximation type results for the fully discrete error, namely,

$$\|u - u_{kh}\|_{L^\infty(I\times\Omega)} \le C|\ln h| \ln \frac{T}{k} \|u - \chi\|_{L^\infty(I\times\Omega)}, \tag{1.2}$$

where u_{kh} denotes the fully discrete solution and χ is an arbitrary element of the finite dimensional space, h is the spatial mesh parameter and k stands for the maximal time step. Such results have only natural assumptions on the problem data and are desirable in many applications, for example in optimal control problems governed by parabolic equations.

Most of the work on pointwise error estimates for parabolic problems were devoted to establishing optimal convergence rates for the error between the exact solution $u(t)$ and the semidiscrete solution $u_h(t)$ that is continuous in time, [3, 4, 5, 6, 20, 21, 23, 24, 30, 32, 33, 41]. The best approximation results for the semidiscrete error $u(t) - u_h(t)$ in $L^\infty(I \times \Omega)$ norm can be found, for example, in [14, 32].

Results on fully discrete pointwise error estimates are much less abundant. Currently, there are several techniques available for obtaining fully discrete error estimates. One popular technique splits the fully discrete error into two parts as $u - u_{kh} = (u - u_h) + (u_h - u_{kh})$. The first part of the error is estimated by the semidiscrete error estimates and the second part of the error is treated by using results from rational approximation of analytic semigroups in Banach spaces. Thus, for example, optimal convergence rates for backward Euler and Crank-Nicolson methods were obtained in [33] (see also [40, Sec. 9] for treatment of general Padé schemes). A similar technique uses a different splitting, $u - u_{kh} = (u - R_h u) + (R_h u - u_{kh})$, where R_h is the Ritz projection. In this approach the first part of the error is treated by elliptic results and the second part of the error satisfies a certain parabolic equation with the right-hand side involving $(u - R_h u)$, which again can be treated by results from rational approximation of analytic semigroups in Banach spaces [19] (see also [40, Thm. 8.6]). For smooth solutions, both

[†]Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA (dmitriy.leykekhman@uconn.edu). The author was partially supported by NSF grant DMS-1522555.

[‡]Lehrstuhl für Mathematische Optimierung, Technische Universität München, Fakultät für Mathematik, Boltzmannstraße 3, 85748 Garching b. München, Germany (vexler@ma.tum.de).

approaches above produce error estimates with optimal convergence rates. However, in many applications these two techniques require unreasonable assumptions on the data, as well as on the regularity of the solution. As a result, the best approximation property (1.2) can not be derived, except for the one-dimensional case [43].

Another approach, that is more direct, is based on the weighted technique. For $N = 2$ and low order time schemes, this technique works rather well and allows one to obtain sharp results. Thus, in [9] (see also [25, Thm. 4.1]) optimal convergence error estimates of the form

$$\|u(t_n) - u_{kh}(t_n)\|_{L^\infty(\Omega)} \le C|\ln h| \left(\ln \frac{t_n}{k}\right)^{\frac{1}{2}} \max_{1\le m\le n} \left(k^q\|\partial_t^q u\|_{L^\infty((0,t_m)\times\Omega)} + h^2\|D^2 u\|_{L^\infty((0,t_m)\times\Omega)}\right),$$

for piecewise constant and piecewise linear time discretizations, i.e. $q = 1$ and $q = 2$, correspondingly, were derived on convex polygonal domains (the result in [9] actually holds even on mildly graded meshes). The best approximation property of the form (1.2) was derived in [28] on convex polygonal domains without any unnatural smoothness requirements. However, for $N = 3$, the weighted technique is much more cumbersome and as of today, there is no three dimensional pointwise best approximation results or optimal error estimates even for backward Euler method.

In this paper for the time discretization we consider discontinuous Galerkin (dG) methods of an arbitrary order. These methods were introduced to parabolic problems in [12] and deeply analyzed in [11]. There are a number of important properties that make dG schemes attractive for temporal discretization of parabolic equations. For example, such schemes allow for a priori error estimates of optimal order with respect to discretization parameters, such as the size of time steps, as well as with respect to the regularity requirements for the solution [8, 9]. Different systematic approaches for a posteriori error estimation and adaptivity developed for finite element discretizations can be adapted for dG temporal discretization of parabolic equations, see, e. g., [37, 38]. Since the trial space allows for discontinuities at the time nodes, the use of different spatial discretizations for each time step can be directly incorporated into the discrete formulation, see, e. g., [37]. Compared to the continuous Galerkin methods, dG schemes are not only A-stable but also strongly A-stable [13]. An efficient and easy to implement approach that avoids complex coefficients, which arise in the equations obtained by a direct decoupling for high order dG schemes, was developed in [29].

Our approach in establishing (1.2) for dG methods is more in the spirit of the work of Palencia [26] and does not require semidiscrete error estimates or even any error splitting. Moreover, it does not require any relationship between the spatial mesh size h and the maximal time step k, which is essential for problems on graded meshes.

Our approach is based on two main tools: The newly established discrete maximal parabolic regularity results [16] for discontinuous Galerkin time schemes and discrete resolvent estimates of the following form:

$$\|(z + \Delta_h)^{-1}\chi\|_{L^\infty(\Omega)} \le \frac{C}{|z|}\|\chi\|_{L^\infty(\Omega)}, \quad \text{for } z \in \mathbb{C} \setminus \Sigma_\gamma, \quad \text{for all } \chi \in \mathbb{V}_h = V_h + iV_h, \tag{1.3}$$

where V_h is the space of continuous Lagrange finite elements and

$$\Sigma_\gamma = \{\, z \in \mathbb{C} \mid |\arg(z)| \le \gamma \,\}, \tag{1.4}$$

for some $\gamma \in (0, \frac{\pi}{2})$ and the constant C that may contain $|\ln h|$ but must be independent of h otherwise. Such a discrete resolvent estimate can be shown directly [1, 2, 17] or by showing stability and smoothing results of the semidiscrete solution operator $E_h(t) = e^{-\Delta_h t}$ [20, 32]. The first approach is preferable since it establishes (1.3) for an arbitrary $\gamma \in (0, \frac{\pi}{2})$, while the second approach via theorem of Hille (see, e.g., Pazy [27], Thm. 2.5.2) only guarantees existence of some $\gamma \in (0, \frac{\pi}{2})$.

In this paper we also establish a local version of the best approximation result (1.2). This result (cf. Theorem 2.2) shows more local behavior of the error at a fixed point. For elliptic problems such estimates are well known (cf. [34, 36, 44]), but for parabolic problems the only result we are aware of is in [28], which is stated for convex polygonal domains without a proof and [15, 18] that are global in time. To obtain this result, in addition to the stability of the Ritz projection in $L^\infty(\Omega)$ norm and the resolvent estimate (1.3), we need the following weighted resolvent estimate

$$\|\sigma^{\frac{N}{2}}(z + \Delta_h)^{-1}\chi\|_{L^2(\Omega)} \le \frac{C|\ln h|}{|z|}\|\sigma^{\frac{N}{2}}\chi\|_{L^2(\Omega)}, \quad \text{for } z \in \mathbb{C} \setminus \Sigma_\gamma, \quad \text{for all } \chi \in \mathbb{V}_h, \tag{1.5}$$

with $\sigma(x) = \sqrt{|x - x_0|^2 + K^2h^2}$. This estimate is established in Theorem 4.1. The estimate (1.5) is somewhat stronger than the corresponding resolvent estimate in L^∞ norm, meaning that (1.3) follows rather easily from (1.5) (modulo logarithmic term $|\ln h|$), but not vice versa.

The rest of the paper is organized as follows. In the next section we describe the discretization method and state our main results. In Section 3, we review some essential elliptic results in weighted norms. Section 4 is devoted to establishing resolvent estimate in weighted norms. In Section 5, we review some results from discrete maximal parabolic regularity. Finally, in Sections 6 and 7, we give proofs of global and interior best approximation properties of the fully discrete solution.

2. Discretization and statement of main results. To introduce the time discontinuous Galerkin discretization for the problem, we partition the interval $(0,T]$ into subintervals $I_m=(t_{m-1},t_m]$ of length $k_m=t_m-t_{m-1}$, where $0=t_0<t_1<\cdots<t_{M-1}<t_M=T$. The maximal and minimal time steps are denoted by $k=\max_m k_m$ and $k_{\min}=\min_m k_m$, respectively. We impose the following conditions on the time mesh (as in [16] or [22]):

(i) There are constants $c,\beta>0$ independent of k such that

$$k_{\min}\geq ck^{\beta}.$$

(ii) There is a constant $\kappa>0$ independent of k such that for all $m=1,2,\dots,M-1$

$$\kappa^{-1}\leq\frac{k_m}{k_{m+1}}\leq\kappa.$$

(iii) It holds $k\leq\frac{1}{4}T$.

The semidiscrete space X_k^q of piecewise polynomial functions in time is defined by

$$X_k^q=\left\{\, u_k\in L^2(I;H_0^1(\Omega))\;\middle|\; u_k|_{I_m}\in\mathcal{P}_q(H_0^1(\Omega)),\ m=1,2,\dots,M\,\right\},$$

where $\mathcal{P}_q(V)$ is the space of polynomial functions of degree q in time with values in a Banach space V. We will employ the following notation for functions in X_k^q

$$u_m^+=\lim_{\varepsilon\to0^+}u(t_m+\varepsilon),\quad u_m^-=\lim_{\varepsilon\to0^+}u(t_m-\varepsilon),\quad [u]_m=u_m^+-u_m^-. \tag{2.1}$$

Next we define the following bilinear form

$$B(u,\varphi)=\sum_{m=1}^{M}\langle\partial_t u,\varphi\rangle_{I_m\times\Omega}+(\nabla u,\nabla\varphi)_{I\times\Omega}+\sum_{m=2}^{M}([u]_{m-1},\varphi_{m-1}^+)_\Omega+(u_0^+,\varphi_0^+)_\Omega, \tag{2.2}$$

where $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_{I_m\times\Omega}$ are the usual L^2 space and space-time inner-products, $\langle\cdot,\cdot\rangle_{I_m\times\Omega}$ is the duality product between $L^2(I_m;H^{-1}(\Omega))$ and $L^2(I_m;H_0^1(\Omega))$. We note, that the first sum vanishes for $u\in X_k^0$. The dG(q) semidiscrete (in time) approximation $u_k\in X_k^q$ of (1.1) is defined as

$$B(u_k,\varphi_k)=(f,\varphi_k)_{I\times\Omega}+(u_0,\varphi_{k,0}^+)_\Omega\quad\text{for all }\ \varphi_k\in X_k^q. \tag{2.3}$$

Rearranging the terms in (2.2), we obtain an equivalent (dual) expression of B:

$$B(u,\varphi)=-\sum_{m=1}^{M}\langle u,\partial_t\varphi\rangle_{I_m\times\Omega}+(\nabla u,\nabla\varphi)_{I\times\Omega}-\sum_{m=1}^{M-1}(u_m^-,[\varphi]_m)_\Omega+(u_M^-,\varphi_M^-)_\Omega. \tag{2.4}$$

Next we define the fully discrete approximation. For $h\in(0,h_0]$; $h_0>0$, let $\mathcal{T}$ denote a quasi-uniform triangulation of Ω with mesh size h, i.e., $\mathcal{T}=\{\tau\}$ is a partition of Ω into cells (triangles or tetrahedrons) τ of diameter h_τ such that for $h=\max_\tau h_\tau$,

$$\operatorname{diam}(\tau)\leq h\leq C|\tau|^{\frac{1}{N}},\quad\forall\tau\in\mathcal{T}.$$

Let V_h be the set of all functions in $H_0^1(\Omega)$ that are polynomials of degree $r\in\mathbb{N}$ on each τ, i.e. V_h is the usual space of conforming finite elements. To obtain the fully discrete approximation we consider the space-time finite element space

$$X_{k,h}^{q,r}=\left\{\, v_{kh}\in L^2(I;V_h)\;\middle|\; v_{kh}|_{I_m}\in\mathcal{P}_q(V_h),\ m=1,2,\dots,M\,\right\},\quad q\geq0,\quad r\geq1. \tag{2.5}$$

We define a fully discrete cG(r)dG(q) solution $u_{kh}\in X_{k,h}^{q,r}$ by

$$B(u_{kh},\varphi_{kh})=(f,\varphi_{kh})_{I\times\Omega}+(u_0,\varphi_{kh}^+)_\Omega\quad\text{for all }\ \varphi_{kh}\in X_{k,h}^{q,r}. \tag{2.6}$$

2.1. Main results. Now we state our main results.

2.1.1. Global pointwise best approximation error estimates. The first result shows best approximation property of $cG(\mathrm{r})dG(\mathrm{q})$ Galerkin solution in $L^\infty(I \times \Omega)$ norm. For $N = 2$ and $q = 0$, $r = 1$, the result can be found in [28] for convex polygonal domains. A similar result showing optimal error estimate is established in [9], Thm. 1.2. We are not aware of any pointwise best approximation type results for $N = 3$.

THEOREM 2.1 (Global best approximation). *Let u and u_{kh} satisfy* (1.1) *and* (2.6) *respectively. Then, there exists a constant C independent of k and h such that*

$$\|u - u_{kh}\|_{L^\infty(I\times\Omega)} \le C \ln\frac{T}{k}|\ln h| \inf_{\chi\in X^{q,r}_{k,h}} \|u-\chi\|_{L^\infty(I\times\Omega)}.$$

The proof of this theorem is given in Section 6.

2.1.2. Interior pointwise best approximation error estimates. For the error at the point x_0 we can obtain a sharper result, that shows more localized behavior of the error at a fixed point. For elliptic problems similar results were obtained in [34, 36]. We denote by $B_d = B_d(x_0)$ the ball of radius d centered at x_0.

THEOREM 2.2 (Interior best approximation). *Let u and u_{kh} satisfy* (1.1) *and* (2.6)*, respectively and let $d > 4h$. Let $\tilde t \in I_m$ with some $m \in \{1, 2, \dots, M\}$ and $\overline{B}_d \subset\subset \Omega$, then there exists a constant C independent of h, k, and d such that*

$$\begin{aligned}|(u-u_{kh})(\tilde t, x_0)| \le C\ln\frac{T}{k}|\ln h| \inf_{\chi\in X^{q,r}_{k,h}} \Bigg\{ &\|u-\chi\|_{L^\infty((0,t_m)\times B_d(x_0))} \\ &+ d^{-\frac{N}{2}}\Big(\|u-\chi\|_{L^\infty((0,t_m);L^2(\Omega))} + h\|\nabla(u-\chi)\|_{L^\infty((0,t_m);L^2(\Omega))}\Big)\Bigg\}.\end{aligned}$$

The proof of this theorem is given in Section 7.

3. Elliptic estimates in weighted norms. In this section we collect some estimates for the finite element discretization of elliptic problems in weighted norms on convex polyhedral domains mainly taken from [17]. These results will be used in the following sections within the proofs of Theorem 2.1 and Theorem 2.2.

Let $x_0 \in \Omega$ be a fixed (but arbitrary) point. Associated with this point we introduce a smoothed Delta function [36, Appendix], which we will denote by $\tilde\delta = \tilde\delta_{x_0}$. This function is supported in one cell, which is denoted by τ_{x_0} and satisfies

$$(\chi, \tilde\delta)_{\tau_{x_0}} = \chi(x_0), \quad \forall \chi \in \mathcal{P}_r(\tau_{x_0}). \tag{3.1}$$

In addition we also have

$$\|\tilde\delta\|_{W^{s,p}(\Omega)} \le Ch^{-s-N(1-\frac{1}{p})}, \quad 1 \le p \le \infty, \quad s = 0, 1. \tag{3.2}$$

Thus in particular $\|\tilde\delta\|_{L^1(\Omega)} \le C$, $\|\tilde\delta\|_{L^2(\Omega)} \le Ch^{-\frac{N}{2}}$, and $\|\tilde\delta\|_{L^\infty(\Omega)} \le Ch^{-N}$. Next we introduce a weight function

$$\sigma(x) = \sqrt{|x-x_0|^2 + K^2h^2}, \tag{3.3}$$

where $K > 0$ is a sufficiently large constant. One can easily check that σ satisfies the following properties:

$$\|\sigma^{-\frac{N}{2}}\|_{L^2(\Omega)} \le C|\ln h|^{\frac{1}{2}}, \tag{3.4a}$$
$$|\nabla\sigma| \le C, \tag{3.4b}$$
$$|\nabla^2\sigma| \le C|\sigma^{-1}| \tag{3.4c}$$
$$\max_{x\in\tau}\sigma \le C\min_{x\in\tau}\sigma, \quad \forall\tau. \tag{3.4d}$$

For the finite element space V_h we will utilize the L^2 projection $P_h\colon L^2(\Omega) \to V_h$ defined by

$$(P_hv, \chi)_\Omega = (v, \chi)_\Omega, \quad \forall \chi \in V_h, \tag{3.5}$$

the Ritz projection $R_h\colon H^1_0(\Omega) \to V_h$ defined by

$$(\nabla R_h v, \nabla \chi)_\Omega = (\nabla v, \nabla \chi)_\Omega, \quad \forall \chi \in V_h, \tag{3.6}$$

and the usual nodal interpolation $i_h\colon C_0(\Omega) \to V_h$. Moreover we introduce the discrete Laplace operator $\Delta_h\colon V_h \to V_h$ defined by

$$(-\Delta_h v_h, \chi)_\Omega = (\nabla v_h, \nabla \chi)_\Omega, \quad \forall \chi \in V_h. \tag{3.7}$$

The following lemma is a superapproximation result in weighted norms.

LEMMA 3.1 (Lemma 2.3 in [17]). *Let $v_h \in V_h$. Then the following estimates hold for any $\alpha, \beta \in \mathbb{R}$ and K large enough:*

$$\|\sigma^\alpha(\operatorname{Id} - i_h)(\sigma^\beta v_h)\|_{L^2(\Omega)} + h\|\sigma^\alpha \nabla(\operatorname{Id} - i_h)(\sigma^\beta v_h)\|_{L^2(\Omega)} \le ch\|\sigma^{\alpha+\beta-1} v_h\|_{L^2(\Omega)}, \tag{3.8}$$

$$\|\sigma^\alpha(\operatorname{Id} - P_h)(\sigma^\beta v_h)\|_{L^2(\Omega)} + h\|\sigma^\alpha \nabla(\operatorname{Id} - P_h)(\sigma^\beta v_h)\|_{L^2(\Omega)} \le ch\|\sigma^{\alpha+\beta-1} v_h\|_{L^2(\Omega)}. \tag{3.9}$$

The next lemma describes a connection between the regularized Delta functional $\tilde{\delta}$ and the weight σ.

LEMMA 3.2. *There holds*

$$\|\sigma^{\frac{N}{2}}\tilde{\delta}\|_{L^2(\Omega)} + h\|\sigma^{\frac{N}{2}}\nabla\tilde{\delta}\|_{L^2(\Omega)} + \|\sigma^{\frac{N}{2}} P_h\tilde{\delta}\|_{L^2(\Omega)} \le C. \tag{3.10}$$

The proof of the above lemma for $N = 2$, for example, can be found in [9] and for $N = 3$ in [17], Lemma 2.4.

The next result shows that the Ritz projection is almost stable in L^∞ norm.

LEMMA 3.3. *There exists a constant $C > 0$ independent on h, such that for any $v \in L^\infty(\Omega) \cap H^1_0(\Omega)$,*

$$\|R_h v\|_{L^\infty(\Omega)} \le C|\ln h|\|v\|_{L^\infty(\Omega)}.$$

For smooth domains such result was established in [35], for polygonal domains in [31], and for convex polyhedral domains in [17, Thm. 3.1]. In the case of smooth domains or for convex polygonal domains the logarithmic factor can be removed for higher than piecewise linear order elements, i.e. $r \ge 2$. The question of log-free stability result for convex polyhedral domains is still open.

Next lemma is rather peculiar and can be thought as weighted Gagliardo-Nirenberg interpolation inequality. The proof is in [17], Lemma 2.5.

LEMMA 3.4. *Let $N = 3$. There exists a constant C independent of K and h such that for any $f \in H^1_0(\Omega)$, any $\alpha, \beta \in \mathbb{R}$ with $\alpha \ge -\frac{1}{2}$ and any $1 \le p \le \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ holds:*

$$\|\sigma^\alpha f\|^2_{L^2(\Omega)} \le C\|\sigma^{\alpha-\beta} f\|_{L^p(\Omega)}\|\sigma^{\alpha+1+\beta}\nabla f\|_{L^{p'}(\Omega)},$$

provided $\|\sigma^{\alpha-\beta} f\|_{L^p(\Omega)}$ and $\|\sigma^{\alpha+1+\beta}\nabla f\|_{L^{p'}(\Omega)}$ are bounded.

4. Weighted resolvent estimates. In this section we will prove weighted resolvent estimates in two and three space dimensions. We will require such estimates to derive smoothing type estimates in the weighted norms in Section 5. Since in this section (only) we will be dealing with complex valued function spaces, we need to modify the definition of the L^2-inner product as

$$(u, v)_\Omega = \int_\Omega u(x)\bar{v}(x)\, dx,$$

where $\bar{v}$ is the complex conjugate of v and the finite element space as $\mathbb{V}_h = V_h + iV_h$.

In the continuous case for Lipschitz domains the following result was shown in [39]: For any $\gamma \in (0, \frac{\pi}{2})$ there exists a constant C independent of z such that

$$\|(z + \Delta)^{-1} v\|_{L^p(\Omega)} \le \frac{C}{1 + |z|}\|v\|_{L^p(\Omega)}, \quad z \in \mathbb{C} \setminus \Sigma_\gamma, \quad 1 \le p \le \infty, \quad v \in L^p(\Omega), \tag{4.1}$$

where Σ_γ is defined by

$$\Sigma_\gamma = \{\, z \in \mathbb{C} \mid |\arg z| \le \gamma \,\}. \tag{4.2}$$

In the finite element setting, it is also known that for any $\gamma \in (0, \frac{\pi}{2})$ there exists a constant C independent of h and z such that

$$\|(z+\Delta_h)^{-1}\chi\|_{L^\infty(\Omega)} \le \frac{C}{1+|z|}\|\chi\|_{L^\infty(\Omega)}, \quad \text{for } z \in \mathbb{C}\setminus\Sigma_\gamma, \quad \text{for all } \chi \in \mathbb{V}_h. \tag{4.3}$$

For smooth domains such result is established in [2] and for convex polyhedral domains with a constant containing $|\ln h|$ in [17]. In [20] the above resolvent result is established for convex polyhedral domains for some $\gamma \in (0, \frac{\pi}{2})$, but with a constant C independent of h.

Our goal in this section is to establish the following resolvent estimate in the weighted norm.

THEOREM 4.1. *For any $\gamma \in (0, \frac{\pi}{2})$, there exists a constant C independent of h and z such that*

$$\|\sigma^{\frac{N}{2}}(z+\Delta_h)^{-1}\chi\|_{L^2(\Omega)} \le \frac{C|\ln h|}{|z|}\|\sigma^{\frac{N}{2}}\chi\|_{L^2(\Omega)}, \quad \textit{for } z \in \mathbb{C}\setminus\Sigma_\gamma,$$

for all $\chi \in \mathbb{V}_h$, where Σ_γ is defined in (4.2).

4.1. Proof of Theorem 4.1 for $N = 2$. For an arbitrary $\chi \in \mathbb{V}_h$ we define

$$u_h = (z+\Delta_h)^{-1}\chi,$$

or equivalently

$$z(u_h, \varphi) - (\nabla u_h, \nabla\varphi) = (\chi, \varphi), \quad \forall \varphi \in \mathbb{V}_h. \tag{4.4}$$

In this section the norm $\|\cdot\|$ will stand for $\|\cdot\|_{L^2(\Omega)}$. To estimate $\|\sigma u_h\|$ we consider the expression

$$\|\sigma\nabla u_h\|^2 = (\nabla(\sigma^2 u_h), \nabla u_h) - 2(\sigma\nabla\sigma u_h, \nabla u_h). \tag{4.5}$$

By taking $\varphi = -P_h(\sigma^2 u_h)$ in (4.4) and adding it to (4.5), we obtain

$$-z\|\sigma u_h\|^2 + \|\sigma\nabla u_h\|^2 = F, \tag{4.6}$$

where

$$F = F_1 + F_2 + F_3 := -(\sigma^2 u_h, \chi) + (\nabla(\sigma^2 u_h - P_h(\sigma^2 u_h)), \nabla u_h) - 2(\sigma\nabla\sigma u_h, \nabla u_h).$$

Since $\gamma \le |\arg z| \le \pi$, this equation is of the form

$$e^{i\alpha}a + b = f, \quad \text{with} \quad a, b > 0, \quad 0 \le |\alpha| \le \pi - \gamma,$$

by multiplying it by $e^{-\frac{i\alpha}{2}}$ and taking real parts, we have

$$a + b \le \left(\cos\left(\frac{\alpha}{2}\right)\right)^{-1}|f| \le \left(\sin\left(\frac{\gamma}{2}\right)\right)^{-1}|f| = C_\gamma|f|.$$

From (4.6) we therefore conclude

$$|z|\|\sigma u_h\|^2 + \|\sigma\nabla u_h\|^2 \le C_\gamma|F|, \quad \text{for } z \in \mathbb{C}\setminus\Sigma_\gamma.$$

Using the Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality we obtain,

$$|F_1| = |(\sigma^2 u_h, \chi)| \le \|\sigma u_h\|\|\sigma\chi\| \le CC_\gamma|z|^{-1}\|\sigma\chi\|^2 + \frac{|z|}{2C_\gamma}\|\sigma u_h\|^2.$$

To estimate F_2 we use Lemma 3.1, the Cauchy-Schwarz and the arithmetic-geometric mean inequalities,

$$|F_2| \le \|\sigma^{-1}\nabla(\sigma^2 u_h - P_h(\sigma^2 u_h))\|\|\sigma\nabla u_h\| \le \frac{1}{4C_\gamma}\|\sigma\nabla u_h\|^2 + CC_\gamma\|u_h\|^2.$$

Finally, using the properties of σ, we obtain

$$|F_3| \le C\|u_h\|\|\sigma\nabla u_h\| \le \frac{1}{4C_\gamma}\|\sigma\nabla u_h\|^2 + CC_\gamma\|u_h\|^2.$$

Combining estimates for $F_i's$ and kicking back, we obtain

$$|z|\|\sigma u_h\|^2 + \|\sigma\nabla u_h\|^2 \le C_\gamma^2\left(|z|^{-1}\|\sigma\chi\|^2 + \|u_h\|^2\right). \tag{4.7}$$

Thus, in order to establish the desired weighted resolvent estimate, we need to show

$$\|u_h\|^2 \le C|\ln h|^2|z|^{-1}\|\sigma\chi\|^2. \tag{4.8}$$

To accomplish that, testing (4.4) with $\varphi = u_h$, we obtain similarly as above

$$|z|\|u_h\|^2 + \|\nabla u_h\|^2 \le C_\gamma|f|, \quad \text{for} \quad z \in \mathbb{C}\setminus\Sigma_\gamma,$$

where $f = (\chi, u_h)$. Using the discrete Sobolev inequality (see [33, Lemma 1.1]),

$$\|v_h\|_{L^\infty(\Omega)} \le C|\ln h|^{\frac{1}{2}}\|\nabla v_h\|_{L^2(\Omega)}, \quad \forall v_h \in V_h,$$

and using the property of σ (3.4a), we obtain

$$\begin{aligned} |z|\|u_h\|^2 + \|\nabla u_h\|^2 &\le C_\gamma\|\sigma\chi\|_{L^2(\Omega)}\|\sigma^{-1}u_h\|_{L^2(\Omega)} \\ &\le C_\gamma\|\sigma\chi\|_{L^2(\Omega)}\|\sigma^{-1}\|_{L^2(\Omega)}\|u_h\|_{L^\infty(\Omega)} \\ &\le C_\gamma|\ln h|\|\sigma\chi\|_{L^2(\Omega)}\|\nabla u_h\|_{L^2(\Omega)} \\ &\le C_\gamma^2|\ln h|^2\|\sigma\chi\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u_h\|^2_{L^2(\Omega)}. \end{aligned}$$

Kicking back $\frac{1}{2}\|\nabla u_h\|^2_{L^2(\Omega)}$, we establish (4.8) and hence Theorem 4.1 in the case of $N = 2$.

4.2. Proof of Theorem 4.1 for $N = 3$. The three dimensional case is more involved and we require some auxiliary results. For a given point $x_0 \in \Omega$, we introduce the adjoint regularized Green's function $G = G^{x_0}(x, \bar{z})$ by

$$G = G^{x_0}(x, \bar{z}) = (\bar{z} + \Delta)^{-1}\tilde{\delta}$$

and its discrete analog $G_h = G_h^{x_0}(x, \bar{z}) \in \mathbb{V}_h$ by

$$G_h = G_h^{x_0}(x, \bar{z}) = (\bar{z} + \Delta_h)^{-1}P_h\tilde{\delta},$$

which we can write in the weak form as

$$z(\varphi, G_h) - (\nabla\varphi, \nabla G_h) = (\varphi, \tilde{\delta}), \quad \forall\varphi \in \mathbb{V}_h. \tag{4.9}$$

From [17] we have the following result.

LEMMA 4.2 ([17]). *Let $G_h \in \mathbb{V}_h$ be defined by* (4.9)*. There holds*

$$\|G_h\|_{L^3(\Omega)} \le C|\ln h|^{\frac{1}{3}}.$$

LEMMA 4.3. *Let $w_h \in \mathbb{V}_h$ be the solution of*

$$z(w_h, \varphi) - (\nabla w_h, \nabla\varphi) = (f, \varphi), \quad \forall\varphi \in \mathbb{V}_h$$

for some $f \in L^{\frac{3}{2}}(\Omega)$. There exists a constant $C > 0$ such that

$$\|w_h\|_{L^\infty(\Omega)} \le C|\ln h|^{\frac{1}{3}}\|f\|_{L^{\frac{3}{2}}(\Omega)}.$$

Proof. There holds

$$w_h(x_0) = z(w_h, G_h) - (\nabla w_h, \nabla G_h) = (f, G_h).$$

Hence,

$$|w_h(x_0)| = |(f, G_h)| \le \|f\|_{L^{\frac{3}{2}}(\Omega)} \|G_h\|_{L^3(\Omega)}.$$

Applying Lemma 4.2 we obtain the result. ∎

LEMMA 4.4. *Let $v_h \in \mathbb{V}_h$ be the solution of*

$$z(v_h, \varphi) - (\nabla v_h, \nabla\varphi) = (f, \varphi), \quad \forall \varphi \in \mathbb{V}_h,$$

and $f \in L^1(\Omega)$. There exists a constant $C > 0$ such that

$$\|v_h\|_{L^3(\Omega)} \le C|\ln h|^{\frac{1}{3}} \|f\|_{L^1(\Omega)}.$$

Proof. We consider a dual solution $w_h \in \mathbb{V}_h$ defined by

$$z(\varphi, w_h) - (\nabla\varphi, \nabla w_h) = (\varphi, v_h|v_h|), \quad \forall \varphi \in \mathbb{V}_h.$$

There holds

$$\|v_h\|^3_{L^3(\Omega)} = z(v_h, w_h) - (\nabla v_h, \nabla w_h) = (f, w_h) \le \|f\|_{L^1(\Omega)} \|w_h\|_{L^\infty(\Omega)}.$$

By Lemma 4.3 that also holds for the adjoint problem, we have

$$\|w_h\|_{L^\infty(\Omega)} \le C|\ln h|^{\frac{1}{3}} \|v_h|v_h|\|_{L^{\frac{3}{2}}(\Omega)} \le C|\ln h|^{\frac{1}{3}} \|v_h\|^2_{L^3(\Omega)}.$$

Thus, we get

$$\|v_h\|^3_{L^3(\Omega)} \le C|\ln h|^{\frac{1}{3}} \|f\|_{L^1(\Omega)} \|v_h\|^2_{L^3(\Omega)}.$$

Canceling $\|v_h\|^2_{L^3(\Omega)}$ completes the proof. ∎

With these results we proceed with the proof of Theorem 4.1 for $N = 3$.

Proof. For an arbitrary $\chi \in \mathbb{V}_h$ we define

$$u_h = (z + \Delta_h)^{-1}\chi.$$

or equivalently

$$z(u_h, \varphi) - (\nabla u_h, \nabla\varphi) = (\chi, \varphi), \quad \forall \varphi \in \mathbb{V}_h. \tag{4.10}$$

To estimate $\|\sigma^{\frac{3}{2}} u_h\|$ we consider the expression

$$\|\sigma^{\frac{3}{2}} \nabla u_h\|^2 = (\nabla(\sigma^3 u_h), \nabla u_h) - 3(\sigma^2 \nabla\sigma u_h, \nabla u_h). \tag{4.11}$$

By taking $\varphi = -P_h(\sigma^3 u_h)$ in (4.10) and adding to (4.11), we obtain

$$-z\|\sigma^{\frac{3}{2}} u_h\|^2 + \|\sigma^{\frac{3}{2}} \nabla u_h\|^2 = F, \tag{4.12}$$

where

$$F = F_1 + F_2 + F_3 := -(P_h(\sigma^3 u_h), \chi) + (\nabla(\sigma^3 u_h - P_h(\sigma^3 u_h)), \nabla u_h) - 3(\sigma^2 \nabla\sigma u_h, \nabla u_h).$$

Since $\gamma \le |\arg z| \le \pi$, this equation is of the form

$$e^{i\alpha} a + b = f, \quad \text{with} \quad a, b > 0, \quad 0 \le |\alpha| \le \pi - \gamma,$$

by multiplying it by $e^{-\frac{i\alpha}{2}}$ and taking real parts, we have

$$a + b \le \left(\cos\left(\frac{\alpha}{2}\right)\right)^{-1} |f| \le \left(\sin\left(\frac{\gamma}{2}\right)\right)^{-1} |f| = C_\gamma |f|.$$

From (4.12) we therefore conclude

$$|z|\|\sigma^{\frac{3}{2}}u_h\|^2 + \|\sigma^{\frac{3}{2}}\nabla u_h\|^2 \le C_\gamma |F|, \quad \text{for } z \in \mathbb{C}\setminus\Sigma_\gamma.$$

Using the Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality we obtain,

$$|F_1| = |(\sigma^3 u_h, \chi)| \le \|\sigma^{\frac{3}{2}}u_h\|\|\sigma^{\frac{3}{2}}\chi\| \le CC_\gamma |z|^{-1}\|\sigma^{\frac{3}{2}}\chi\|^2 + \frac{|z|}{2C_\gamma}\|\sigma^{\frac{3}{2}}u_h\|^2.$$

To estimate F_2 we use Lemma 3.1, the Cauchy-Schwarz and the arithmetic-geometric mean inequalities,

$$|F_2| \le \|\sigma^{-\frac{3}{2}}\nabla(\sigma^3 u_h - P_h(\sigma^3 u_h))\|\|\sigma^{\frac{3}{2}}\nabla u_h\| \le \frac{1}{4C_\gamma}\|\sigma^{\frac{3}{2}}\nabla u_h\|^2 + CC_\gamma\|\sigma^{\frac{1}{2}}u_h\|^2.$$

Finally, using the properties of σ, we obtain

$$|F_3| \le C\|\sigma^{\frac{1}{2}}u_h\|\|\sigma^{\frac{3}{2}}\nabla u_h\| \le \frac{1}{4C_\gamma}\|\sigma^{\frac{3}{2}}\nabla u_h\|^2 + CC_\gamma\|\sigma^{\frac{1}{2}}u_h\|^2.$$

Combining the estimates for $F_i's$ and kicking back, we obtain

$$|z|\|\sigma^{\frac{3}{2}}u_h\|^2 + \|\sigma^{\frac{3}{2}}\nabla u_h\|^2 \le C\left(|z|^{-1}\|\sigma^{\frac{3}{2}}\chi\|^2 + \|\sigma^{\frac{1}{2}}u_h\|^2\right). \tag{4.13}$$

Thus, in order to establish the desired weighted resolvent estimate, we need to show

$$\|\sigma^{\frac{1}{2}}u_h\|^2 \le C|\ln h|^2|z|^{-1}\|\sigma^{\frac{3}{2}}\chi\|^2. \tag{4.14}$$

To accomplish that, we consider the expression

$$-z\|\sigma^{\frac{1}{2}}u_h\|^2 + \|\sigma^{\frac{1}{2}}\nabla u_h\|^2 = -z(u_h, \sigma u_h) + (\nabla u_h, \nabla(\sigma u_h)) - (\nabla u_h, \nabla\sigma u_h).$$

Testing (4.10) with $\varphi = P_h(\sigma u_h)$ we obtain similarly as above

$$|z|\|\sigma^{\frac{1}{2}}u_h\|^2 + \|\sigma^{\frac{1}{2}}\nabla u_h\|^2 \le C_\gamma|f|, \quad \text{for} \quad z \in \mathbb{C}\setminus\Sigma_\gamma,$$

where

$$f = f_1 + f_2 + f_3 := -(P_h(\sigma u_h), \chi) + (\nabla(\sigma u_h - P_h(\sigma u_h)), \nabla u_h) - (\nabla\sigma u_h, \nabla u_h).$$

Using the Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality, we obtain

$$|f_1| = |(\sigma u_h, \chi)| \le \|\sigma^{-\frac{1}{2}}u_h\|\|\sigma^{\frac{3}{2}}\chi\| \le \frac{1}{2}\|\sigma^{-\frac{1}{2}}u_h\|^2 + \frac{1}{2}\|\sigma^{\frac{3}{2}}\chi\|^2.$$

To estimate f_2 we use Lemma 3.1, the Cauchy-Schwarz and the arithmetic-geometric mean inequalities,

$$|f_2| \le \|\sigma^{-\frac{1}{2}}\nabla(\sigma u_h - P_h(\sigma u_h))\|\|\sigma^{\frac{1}{2}}\nabla u_h\| \le \frac{1}{4C_\gamma}\|\sigma^{\frac{1}{2}}\nabla u_h\|^2 + CC_\gamma\|\sigma^{-\frac{1}{2}}u_h\|^2.$$

Finally, using the properties of σ, we obtain

$$|f_3| \le C\|\sigma^{-\frac{1}{2}}u_h\|\|\sigma^{\frac{1}{2}}\nabla u_h\| \le \frac{1}{4C_\gamma}\|\sigma^{\frac{1}{2}}\nabla u_h\|^2 + CC_\gamma\|\sigma^{-\frac{1}{2}}u_h\|^2.$$

Combining estimates for $f_i's$ and kicking back, we obtain

$$|z|\|\sigma^{\frac{1}{2}}u_h\|^2 + \|\sigma^{\frac{1}{2}}\nabla u_h\|^2 \le C\left(\|\sigma^{-\frac{1}{2}}u_h\|^2 + \|\sigma^{\frac{3}{2}}\chi\|^2\right). \tag{4.15}$$

To estimate $\|\sigma^{-\frac{1}{2}}u_h\|$ we use Lemma 3.4 with $\alpha = \beta = -\frac{1}{2}$ and $p = 3$, to obtain

$$\|\sigma^{-\frac{1}{2}}u_h\| \le C\|u_h\|^{\frac{1}{2}}_{L^3(\Omega)}\|\nabla u_h\|^{\frac{1}{2}}_{L^{\frac{3}{2}}(\Omega)}. \tag{4.16}$$

Using Lemma 4.4, we have

$$\|u_h\|_{L^3(\Omega)} \le C|\ln h|^{\frac{1}{3}}\|\chi\|_{L^1(\Omega)} \le C|\ln h|^{\frac{1}{3}}\|\sigma^{-\frac{3}{2}}\|\|\sigma^{\frac{3}{2}}\chi\| \le C|\ln h|^{\frac{5}{6}}\|\sigma^{\frac{3}{2}}\chi\|.$$

To estimate $\|\nabla u_h\|_{L^{\frac{3}{2}}(\Omega)}$ we proceed by the Hölder inequality

$$\|\nabla u_h\|_{L^{\frac{3}{2}}(\Omega)} \le C|\ln h|^{\frac{1}{6}}\|\sigma^{\frac{1}{2}}\nabla u_h\|_{L^2(\Omega)}. \tag{4.17}$$

Thus, using (4.15) and the above estimates, we have

$$\begin{aligned}|z|\|\sigma^{\frac{1}{2}}u_h\|^2 + \|\sigma^{\frac{1}{2}}\nabla u_h\|^2 &\le C\left(\|u_h\|_{L^3(\Omega)}\|\nabla u_h\|_{L^{\frac{3}{2}}(\Omega)} + \|\sigma^{\frac{3}{2}}\chi\|^2\right)\\ &\le C\left(|\ln h|\|\sigma^{\frac{1}{2}}\nabla u_h\|\|\sigma^{\frac{3}{2}}\chi\| + \|\sigma^{\frac{3}{2}}\chi\|^2\right)\\ &\le C|\ln h|^2\|\sigma^{\frac{3}{2}}\chi\|^2 + \frac{1}{2}\|\sigma^{\frac{1}{2}}\nabla u_h\|^2.\end{aligned}$$

Kicking back $\|\sigma^{\frac{1}{2}}\nabla u_h\|^2$, we finally obtain

$$\|\sigma^{\frac{1}{2}}u_h\|^2 \le C|\ln h|^2|z|^{-1}\|\sigma^{\frac{3}{2}}\chi\|^2,$$

which shows (4.14) and hence the theorem. □

5. Maximal parabolic and smoothing estimates. In this section we state some smoothing and stability results for homogeneous and inhomogeneous problems that are central in establishing our main results. Since we apply the following results for different norms on V_h, namely, for $L^p(\Omega)$ and weighted $L^2(\Omega)$ norms, we state them for a general norm $|||\cdot|||$.

Let $|||\cdot|||$ be a norm on V_h (extended in a straightforward way to a norm on $\mathbb{V}_h$) such that for some $\gamma \in (0, \frac{\pi}{2})$ the following resolvent estimate holds,

$$\left|\left|\left|(z+\Delta_h)^{-1}\chi\right|\right|\right| \le \frac{M_h}{|z|}|||\chi|||, \quad \text{for } z \in \mathbb{C}\setminus\Sigma_\gamma, \tag{5.1}$$

for all $\chi \in \mathbb{V}_h$, where Σ_γ is defined in (4.2) and the constant M_h is independent of z.

This assumption is fulfilled for $|||\cdot||| = \|\cdot\|_{L^p(\Omega)}$, $1 \le p \le \infty$, with a constant $M_h \le C$ independent of h, see [21], and for $|||\cdot||| = \|\sigma^{\frac{N}{2}}\cdot\|_{L^2(\Omega)}$ with $M_h \le C|\ln h|$, see Theorem 4.1.

5.1. Smoothing estimates for the homogeneous problem in Banach spaces. First, we consider the homogeneous heat equation (1.1), i.e. with $f = 0$ and its discrete approximation $u_{kh} \in X^{q,r}_{k,h}$ defined by

$$B(u_{kh}, \varphi_{kh}) = (u_0, \varphi^+_{kh,0}) \quad \forall \varphi_{kh} \in X^{q,r}_{k,h}. \tag{5.2}$$

The first result is a smoothing type estimate, see [16, Theorem 13], cf. also [10, Thmeorem 5.1] for the case of the L^2 norm.

LEMMA 5.1 (Fully discrete homogeneous smoothing estimate). *Let $|||\cdot|||$ be a norm on V_h fulfilling the resolvent estimate* (5.1)*. Let u_{kh} be the solution of* (5.2)*. Then, there exists a constant C independent of k and h such that*

$$\sup_{t\in I_m}|||\partial_t u_{kh}(t)||| + \sup_{t\in I_m}|||\Delta_h u_{kh}(t)||| + k_m^{-1}|||[u_{kh}]_{m-1}||| \le \frac{CM_h}{t_m}|||P_h u_0|||,$$

for $m = 1, 2, \dots, M$. For $m = 1$ the jump term is understood as $[u_{kh}]_0 = u^+_{kh,0} - P_h u_0$.

For the proofs of Theorem 2.1 and Theorem 2.2, we will need an additional stability result, which is also formulated for a general norm $|||\cdot|||$ fulfilling (5.1).

LEMMA 5.2. *Let $|||\cdot|||$ be a norm on V_h fulfilling the resolvent estimate* (5.1)*. Let u_{kh} be the solution of* (5.2)*. Then there exists a constant C independent of k and h such that*

$$\sum_{m=1}^{M}\left(\int_{I_m}|||\partial_t u_{kh}(t)|||dt + \int_{I_m}|||\Delta_h u_{kh}(t)|||dt + |||[u_{kh}]_{m-1}|||\right) \le CM_h \ln\frac{T}{k}|||P_h u_0|||.$$

For $m = 1$ *the jump term is understood as* $[u_{kh}]_0 = u^+_{kh,0} - P_h u_0$.

Proof. Using the above smoothing result, we have

$$\begin{aligned}
&\sum_{m=1}^{M} \left(\int_{I_m} |||\partial_t u_{kh}(t)||| dt + \int_{I_m} |||\Delta_h u_{kh}(t)||| dt + |||[u_{kh}]_{m-1}||| \right) \\
&\le \sum_{m=1}^{M} k_m \left(\sup_{t \in I_m} |||\partial_t u_{kh}(t)||| + \sup_{t \in I_m} |||\Delta_h u_{kh}(t)||| + k_m^{-1} |||[u_{kh}]_{m-1}||| \right) \\
&\le C M_h \sum_{m=1}^{M} \frac{k_m}{t_m} |||P_h u_0||| \le C M_h \ln \frac{T}{k} |||P_h u_0|||,
\end{aligned}$$

where in the last step we used that $\sum_{m=1}^{M} \frac{k_m}{t_m} \le C \ln \frac{T}{k}$. ☐

5.2. Discrete maximal parabolic estimates for the inhomogeneous problem in Banach spaces. Now, we consider the inhomogeneous heat equation (1.1), with $u_0 = 0$ and its discrete approximation $u_{kh} \in X^{q,r}_{k,h}$ defined by

$$B(u_{kh}, \varphi_{kh}) = (f, \varphi_{kh}), \quad \forall \varphi_{kh} \in X^{q,r}_{k,h}. \tag{5.3}$$

The following discrete maximal parabolic regularity result is taken from [16, Theorem 14].

LEMMA 5.3 (Discrete maximal parabolic regularity). *Let* $|||\cdot|||$ *be a norm on* V_h *fulfilling the resolvent estimate* (5.1) *and let* $1 \le s \le \infty$*. Let* u_{kh} *be a solution of* (5.3)*. Then, there exists a constant* C *independent of* k *and* h *such that*

$$\begin{aligned}
&\left(\sum_{m=1}^{M} \int_{I_m} |||\partial_t u_{kh}(t)|||^s dt \right)^{\frac{1}{s}} + \left(\sum_{m=1}^{M} \int_{I_m} |||\Delta_h u_{kh}(t)|||^s dt \right)^{\frac{1}{s}} + \left(\sum_{m=1}^{M} k_m \left|\left|\left| k_m^{-1} [u_{kh}]_{m-1} \right|\right|\right|^s \right)^{\frac{1}{s}} \\
&\qquad\qquad \le C M_h \ln \frac{T}{k} \left(\int_I |||P_h f(t)|||^s dt \right)^{\frac{1}{s}},
\end{aligned}$$

with obvious notation change in the case of $s = \infty$*. For* $m = 1$ *the jump term is understood as* $[u_{kh}]_0 = u^+_{kh,0}$.

REMARK 5.4. *As mentioned above the assumption* (5.1) *is fulfilled for* $|||\cdot||| = \|\cdot\|_{L^p(\Omega)}$ *and any* $1 \le p \le \infty$ *with* $M_h \le C$ *and for* $|||\cdot||| = \|\sigma^{\frac{N}{2}} \cdot\|_{L^2(\Omega)}$ *with* $M_h \le C |\ln h|$*. Therefore the results of Lemma 5.1, Lemma 5.2, and Lemma 5.3 are fulfilled for these two choices of norms with the corresponding constants* M_h.

6. Proof of Theorem 2.1. Let $\tilde{t} \in (0, T]$ and let $x_0 \in \Omega$ be an arbitrary but fixed point. Without loss of generality we assume $\tilde{t} \in (t_{M-1}, T]$. We consider two cases: $\tilde{t} = T$ and $t_{M-1} < \tilde{t} < T$.

Case 1, $\boldsymbol{\tilde{t} = T}$**:** To establish our result we will estimate $u_{kh}(T, x_0)$ by using a duality argument. First, we define g to be a solution to the following backward parabolic problem

$$\begin{aligned}
-\partial_t g(t,x) - \Delta g(t,x) &= 0 \qquad && (t,x) \in I \times \Omega, \\
g(t,x) &= 0, && (t,x) \in I \times \partial\Omega, \\
g(T,x) &= \tilde{\delta}_{x_0}, && x \in \Omega,
\end{aligned} \tag{6.1}$$

where $\tilde{\delta} = \tilde{\delta}_{x_0}$ is the smoothed Dirac function introduced in (3.1). Let $g_{kh} \in X^{q,r}_{k,h}$ be the corresponding cG(r)dG(q) solution defined by

$$B(\varphi_{kh}, g_{kh}) = \varphi_{kh}(T, x_0) \quad \forall \varphi_{kh} \in X^{q,r}_{k,h}. \tag{6.2}$$

Then using that cG(r)dG(q) method is consistent, we have

$$\begin{aligned}
u_{kh}(T, x_0) &= B(u_{kh}, g_{kh}) = B(u, g_{kh}) \\
&= -\sum_{m=1}^{M} (u, \partial_t g_{kh})_{I_m \times \Omega} + (\nabla u, \nabla g_{kh})_{I \times \Omega} - \sum_{m=1}^{M-1} (u_m, [g_{kh}]_m)_\Omega + (u(T), g^-_{kh,M})_\Omega \\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{6.3}$$

Using the Hölder inequality we have

$$\begin{aligned} J_1 &\le \sum_{m=1}^{M} \|u\|_{L^\infty(I_m\times\Omega)} \|\partial_t g_{kh}\|_{L^1(I_m;L^1(\Omega))} \\ &\le \|u\|_{L^\infty(I\times\Omega)} \sum_{m=1}^{M} \|\partial_t g_{kh}\|_{L^1(I_m;L^1(\Omega))}. \end{aligned} \tag{6.4}$$

For J_2 we obtain using the stability of the Ritz projection in $L^\infty(\Omega)$ norm on polygonal and polyhedral domains, see Lemma 3.3,

$$\begin{aligned} J_2 &= (\nabla R_h u, \nabla g_{kh})_{I\times\Omega} = -(R_h u, \Delta_h g_{kh})_{I\times\Omega} \\ &\le \|R_h u\|_{L^\infty(I\times\Omega)} \|\Delta_h g_{kh}\|_{L^1(I;L^1(\Omega))} \\ &\le C|\ln h| \|u\|_{L^\infty(I\times\Omega)} \|\Delta_h g_{kh}\|_{L^1(I;L^1(\Omega))} \end{aligned} \tag{6.5}$$

For J_3 and J_4 we obtain

$$\begin{aligned} J_3 &\le \sum_{m=1}^{M-1} \|u_m\|_{L^\infty(\Omega)} \|[g_{kh}]_m\|_{L^1(\Omega)} \le \|u\|_{L^\infty(I\times\Omega)} \sum_{m=1}^{M-1} \|[g_{kh}]_m\|_{L^1(\Omega)}, \\ J_4 &\le \|u(T)\|_{L^\infty(\Omega)} \|g_{kh,M}^-\|_{L^1(\Omega)} \le \|u\|_{L^\infty(I\times\Omega)} \|g_{kh,M}^-\|_{L^1(\Omega)}. \end{aligned} \tag{6.6}$$

Combining the estimates for J_1, J_2, J_3, and J_4 and applying Lemma 5.2 with $|||\cdot||| = \|\cdot\|_{L^1(\Omega)}$ and $M_h \le C$, cf. Remark 5.4, we have

$$\begin{aligned} |u_{kh}(T,x_0)| &\le C|\ln h| \|u\|_{L^\infty(I\times\Omega)} \Bigg(\sum_{m=1}^{M} \|\partial_t g_{kh}\|_{L^1(I_m;L^1(\Omega))} + \|\Delta_h g_{kh}\|_{L^1(I;L^1(\Omega))} \\ &\qquad + \sum_{m=1}^{M-1} \|[g_{kh}]_m\|_{L^1(\Omega)} + \|g_{kh,M}^-\|_{L^1(\Omega)} \Bigg) \\ &\le C|\ln h| \ln\frac{T}{k} \|u\|_{L^\infty(I\times\Omega)} \|P_h\tilde\delta\|_{L^1(\Omega)} \\ &\le C|\ln h| \ln\frac{T}{k} \|u\|_{L^\infty(I\times\Omega)}, \end{aligned}$$

where in the last step we used the stability of the L^2 projection P_h with respect to the $L^1(\Omega)$ norm, see, e. g., [7] and the fact that $\|\tilde\delta\|_{L^1(\Omega)} \le C$.

Using that the cG(r)dG(q) method is invariant on $X_{k,h}^{q,r}$, by replacing u and u_{kh} with $u-\chi$ and $u_{kh}-\chi$ for any $\chi \in X_{k,h}^{q,r}$, and using the triangle inequality we obtain

$$|u(T,x_0) - u_{kh}(T,x_0)| \le C \ln\frac{T}{k} |\ln h| \inf_{\chi\in X_{k,h}^{q,r}} \|u-\chi\|_{L^\infty(I\times\Omega)}.$$

Case 2, $t_{M-1} < \tilde t < T$:

In this case we consider the following regularized Green's function

$$\begin{aligned} -\partial_t \tilde g(t,x) - \Delta \tilde g(t,x) &= \tilde\delta_{x_0}(x)\tilde\theta(t) \qquad && (t,x) \in I\times\Omega, \\ \tilde g(t,x) &= 0, && (t,x) \in I\times\partial\Omega, \\ \tilde g(T,x) &= 0, && x\in\Omega, \end{aligned} \tag{6.7}$$

where $\tilde\theta \in C^1(\bar I)$ is the regularized Delta function in time with properties

$$\operatorname{supp}\tilde\theta \subset (t_{M-1},T), \quad \|\tilde\theta\|_{L^1(I_M)} \le C$$

and

$$(\tilde\theta, \varphi_k)_{I_M} = \varphi_k(\tilde t), \quad \forall \varphi_k \in \mathcal{P}_q(I_M).$$

Let $\tilde{g}_{kh}$ be cG(r)dG(q) approximation of $\tilde{g}$, i.e.

$$B(\varphi_{kh}, \tilde{g} - \tilde{g}_{kh}) = 0 \quad \forall \varphi_{kh} \in X_{k,h}^{q,r}.$$

Then, using that cG(r)dG(q) method is consistent, we have

$$\begin{aligned} u_{kh}(\tilde{t}, x_0) &= (u_{kh}, \tilde{\delta}_{x_0}\tilde{\theta}) = B(u_{kh}, \tilde{g}) = B(u_{kh}, \tilde{g}_{kh}) = B(u, \tilde{g}_{kh}) \\ &= -\sum_{m=1}^{M} (u, \partial_t \tilde{g}_{kh})_{I_m \times \Omega} + (\nabla u, \nabla \tilde{g}_{kh})_{I \times \Omega} - \sum_{m=1}^{M} (u_m, [\tilde{g}_{kh}]_m)_\Omega, \end{aligned}$$

where in the sum with jumps we included the last term by setting $\tilde{g}_{kh,M+1} = 0$ and defining consequently $[\tilde{g}_{kh}]_M = -\tilde{g}_{kh,M}$. Similarly to the estimates of J_1, J_2, J_3 above, using the stability of the Ritz projection in L^∞ norm on polyhedral domains, see Lemma 3.3, we have

$$\begin{aligned} u_{kh}(\tilde{t}, x_0) &= -\sum_{m=1}^{M} (u, \partial_t \tilde{g}_{kh})_{I \times \Omega} + (\nabla u, \nabla \tilde{g}_{kh})_{I \times \Omega} - \sum_{m=1}^{M} (u_m, [\tilde{g}_{kh}]_m)_\Omega \\ &\le C|\ln h| \|u\|_{L^\infty(I\times\Omega)} \left(\sum_{m=1}^{M} \|\partial_t \tilde{g}_{kh}\|_{L^1(I_m;L^1(\Omega))} + \|\Delta_h \tilde{g}_{kh}\|_{L^1(I;L^1(\Omega))} + \sum_{m=1}^{M} \|[\tilde{g}_{kh}]_m\|_{L^1(\Omega)} \right). \end{aligned}$$

Using the discrete maximal parabolic regularity result from Lemma 5.3 with $|||\cdot||| = \|\cdot\|_{L^1(\Omega)}$ and $M_h \le C$, cf. Remark 5.4, we obtain

$$u_{kh}(\tilde{t}, x_0) \le C \ln \frac{T}{k} |\ln h| \|u\|_{L^\infty(I\times\Omega)} \|P_h \tilde{\delta}_{x_0}\|_{L^1(\Omega)} \|\tilde{\theta}\|_{L^1(I_M)} \le C \ln \frac{T}{k} |\ln h| \|u\|_{L^\infty(I\times\Omega)}.$$

As in the first case this implies

$$|u(\tilde{t}, x_0) - u_{kh}(\tilde{t}, x_0)| \le C \ln \frac{T}{k} |\ln h| \inf_{\chi \in X_{k,h}^{q,r}} \|u - \chi\|_{L^\infty(I\times\Omega)}.$$

This completes the proof of the theorem.

7. Proof of Theorem 2.2. To obtain the interior estimate we introduce a smooth cut-off function ω with the properties that

$$\omega(x) \equiv 1, \quad x \in B_d \tag{7.1a}$$
$$\omega(x) \equiv 0, \quad x \in \Omega \setminus B_{2d} \tag{7.1b}$$
$$|\nabla \omega| \le Cd^{-1}, \quad |\nabla^2 \omega| \le Cd^{-2}, \tag{7.1c}$$

where $B_d = B_d(x_0)$ is a ball of radius d centered at x_0.

As in the proof of Theorem 2.1 we consider two cases: $\tilde{t} = T$ and $t_{M-1} < \tilde{t} < T$. In the first case we obtain

$$u_{kh}(T, x_0) = B(u_{kh}, g_{kh}) = B(u, g_{kh}) = B(\omega u, g_{kh}) + B((1-\omega)u, g_{kh}), \tag{7.2}$$

where g is the solution of (6.1) and $g_{kh} \in X_{k,h}^{q,r}$ is the solution of (6.2). The first term can be estimated using the global result from Theorem 2.1. To this end we introduce $\tilde{u} = \omega u$ and the cG(r)dG(q) solution $\tilde{u}_{kh} \in X_{k,h}^{q,r}$ defined by

$$B(\tilde{u}_{kh} - \tilde{u}, \varphi_{kh}) = 0 \quad \text{for all } \varphi_{kh} \in X_{k,h}^{q,r}.$$

There holds

$$B(\tilde{u}, g_{kh}) = B(\tilde{u}_{kh}, g_{kh}) = \tilde{u}_{kh}(T, x_0) \le C \ln \frac{T}{k} |\ln h| \|\tilde{u}\|_{L^\infty(I\times\Omega)} \le C \ln \frac{T}{k} |\ln h| \|u\|_{L^\infty(I\times B_{2d})}.$$

This results in

$$|u_{kh}(T, x_0)| \le C \ln \frac{T}{k} |\ln h| \|u\|_{L^\infty(I\times B_{2d})} + B((1-\omega)u, g_{kh}). \tag{7.3}$$

It remains to estimate the term $B((1-\omega)u, g_{kh})$. Using the dual expression (2.4) of the bilinear form B we obtain

$$
\begin{aligned}
B((1-\omega)u, g_{kh}) &= -\sum_{m=1}^{M}((1-\omega)u, \partial_t g_{kh})_{I_m\times\Omega} + (\nabla((1-\omega)u), \nabla g_{kh})_{I\times\Omega}\\
&\quad -\sum_{m=1}^{M-1}((1-\omega)u_m, [g_{kh}]_m)_\Omega + ((1-\omega)u(T), g_{kh,M}^-)_\Omega\\
&= -\sum_{m=1}^{M}(\sigma^{-\frac{N}{2}}(1-\omega)u, \sigma^{\frac{N}{2}}\partial_t g_{kh})_{I_m\times\Omega} + (\nabla((1-\omega)u), \nabla g_{kh})_{I\times\Omega}\\
&\quad -\sum_{m=1}^{M-1}(\sigma^{-\frac{N}{2}}(1-\omega)u_m, \sigma^{\frac{N}{2}}[g_{kh}]_m)_\Omega + (\sigma^{-\frac{N}{2}}(1-\omega)u(T), \sigma^{\frac{N}{2}}g_{kh,M}^-)_\Omega\\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned}
\tag{7.4}
$$

For J_1, using that $\sigma^{-\frac{N}{2}} \le Cd^{-\frac{N}{2}}$ on $\operatorname{supp}(1-\omega)\subset\Omega\setminus B_d$ and $(1-\omega)\le 1$, we obtain

$$
\begin{aligned}
J_1 &\le \|\sigma^{-\frac{N}{2}}(1-\omega)u\|_{L^\infty(I;L^2(\Omega))}\sum_{m=1}^{M}\|\sigma^{\frac{N}{2}}\partial_t g_{kh}\|_{L^1(I_m;L^2(\Omega))}\\
&\le Cd^{-\frac{N}{2}}\|u\|_{L^\infty(I;L^2(\Omega))}\sum_{m=1}^{M}\|\sigma^{\frac{N}{2}}\partial_t g_{kh}\|_{L^1(I_m;L^2(\Omega))}.
\end{aligned}
\tag{7.5}
$$

To estimate J_2, we define $\psi = (1-\omega)u$ and proceed using the Ritz projection R_h defined by (3.6). There holds

$$
\begin{aligned}
(\nabla\psi(t), \nabla g_{kh}(t))_\Omega &= (\nabla R_h\psi(t), \nabla g_{kh}(t))_\Omega = -(R_h\psi(t), \Delta_h g_{kh}(t))_\Omega\\
&= -(R_h\psi(t), \Delta_h g_{kh}(t))_{B_{d/2}} - (R_h\psi(t), \Delta_h g_{kh}(t))_{\Omega\setminus B_{d/2}}\\
&\le \|R_h\psi(t)\|_{L^\infty(B_{d/2})}\|\Delta_h g_{kh}(t)\|_{L^1(B_{d/2})}\\
&\qquad + Cd^{-\frac{N}{2}}\|R_h\psi(t)\|_{L^2(\Omega\setminus B_{d/2})}\|\sigma^{\frac{N}{2}}\Delta_h g_{kh}(t)\|_{L^2(\Omega\setminus B_{d/2})}\\
&\le \|R_h\psi(t)\|_{L^\infty(B_{d/2})}\|\Delta_h g_{kh}(t)\|_{L^1(\Omega)} + Cd^{-\frac{N}{2}}\|R_h\psi(t)\|_{L^2(\Omega)}\|\sigma^{\frac{N}{2}}\Delta_h g_{kh}(t)\|_{L^2(\Omega)},
\end{aligned}
$$

where we used $\sigma^{-\frac{N}{2}} \le Cd^{-\frac{N}{2}}$ on $\Omega\setminus B_{d/2}$. In the interior pointwise error estimates [36, Thm. 1.1] with $F\equiv 0$, choosing $\chi = 0$, $s = 0$, $q = 2$ and using the triangle inequality and the fact that $\operatorname{supp}\psi(t)\subset\Omega\setminus B_d$, we have

$$
\|R_h\psi(t)\|_{L^\infty(B_{d/2})} \le C|\ln h|\|\psi(t)\|_{L^\infty(B_d)} + Cd^{-\frac{N}{2}}\|R_h\psi(t)\|_{L^2(\Omega)} = Cd^{-\frac{N}{2}}\|R_h\psi(t)\|_{L^2(\Omega)}.
$$

Using a standard elliptic estimate and recalling $\psi = (1-\omega)u$ we have

$$
\begin{aligned}
\|R_h\psi(t)\|_{L^2(\Omega)} &\le \|\psi(t)\|_{L^2(\Omega)} + \|\psi(t) - R_h\psi(t)\|_{L^2(\Omega)}\\
&\le \|\psi(t)\|_{L^2(\Omega)} + ch\|\nabla\psi(t)\|_{L^2(\Omega)}\\
&\le \|u(t)\|_{L^2(\Omega)} + ch\|(1-\omega)\nabla u(t) - \nabla\omega u(t)\|_{L^2(\Omega)}\\
&\le c\|u(t)\|_{L^2(\Omega)} + ch\|\nabla u(t)\|_{L^2(\Omega)},
\end{aligned}
$$

where in the last step we used $|\nabla\omega| \le Cd^{-1} \le Ch^{-1}$.

Therefore we obtain

$$
(\nabla\psi(t), \nabla g_{kh}(t))_\Omega \le Cd^{-\frac{N}{2}}\left(\|u(t)\|_{L^2(\Omega)} + ch\|\nabla u(t)\|_{L^2(\Omega)}\right)\left(\|\Delta_h g_{kh}(t)\|_{L^1(\Omega)} + \|\sigma^{\frac{N}{2}}\Delta_h g_{kh}(t)\|_{L^2(\Omega)}\right).
$$

This results in

$$
J_2 \le Cd^{-\frac{N}{2}}\left(\|u\|_{L^\infty(I;L^2(\Omega))} + ch\|\nabla u\|_{L^\infty(I;L^2(\Omega))}\right)\left(\|\Delta_h g_{kh}\|_{L^1(I;L^1(\Omega))} + \|\sigma^{\frac{N}{2}}\Delta_h g_{kh}\|_{L^1(I;L^2(\Omega))}\right).
\tag{7.6}
$$

For J_3, similarly to J_1 we obtain

$$\begin{aligned} J_3 &\le \|\sigma^{-\frac{N}{2}}(1-\omega)u\|_{L^\infty(I;L^2(\Omega))} \sum_{m=1}^{M-1} \|\sigma^{\frac{N}{2}}[g_{kh}]_m\|_{L^2(\Omega)} \\ &\le Cd^{-\frac{N}{2}}\|u\|_{L^\infty(I;L^2(\Omega))} \sum_{m=1}^{M-1} \|\sigma^{\frac{N}{2}}[g_{kh}]_m\|_{L^2(\Omega)}. \end{aligned} \tag{7.7}$$

Finally,

$$J_4 \le Cd^{-\frac{N}{2}}\|u\|_{L^\infty(I;L^2(\Omega))}\|\sigma^{\frac{N}{2}} g^-_{kh,M}\|_{L^2(\Omega)}. \tag{7.8}$$

Combining the estimates for J_1, J_2, J_3, and J_4, we have

$$\begin{aligned} B((1-\omega)v, g_{kh}) \le Cd^{-\frac{N}{2}} &\left(\|u\|_{L^\infty(I;L^2(\Omega))} + ch\|\nabla u\|_{L^\infty(I;L^2(\Omega))}\right) \\ \times &\left(\sum_{m=1}^{M} \|\sigma^{\frac{N}{2}}\partial_t g_{kh}\|_{L^1(I_m;L^2(\Omega))} + \|\Delta_h g_{kh}\|_{L^1(I;L^1(\Omega))} + \|\sigma^{\frac{N}{2}}\Delta_h g_{kh}\|_{L^1(I;L^2(\Omega))} \right. \\ &\left. + \sum_{m=1}^{M-1} \|\sigma^{\frac{N}{2}}[g_{kh}]_m\|_{L^2(\Omega)} + \|\sigma^{\frac{N}{2}} g^-_{kh,M}\|_{L^2(\Omega)} \right). \end{aligned}$$

For the term $\|\Delta_h g_{kh}\|_{L^1(I;L^1(\Omega))}$ we apply Lemma 5.2 with $|||\cdot||| = \|\cdot\|_{L^1(\Omega)}$ and $M_h \le C$ and for all weighted terms with $|||\cdot||| = \|\sigma^{\frac{N}{2}}(\cdot)\|_{L^2(\Omega)}$ and $M_h \le C|\ln h|$, cf. Remark 5.4, resulting in

$$\begin{aligned} B((1-\omega)v, g_{kh}) &\le Cd^{-\frac{N}{2}} \ln\frac{T}{k}|\ln h| \left(\|u\|_{L^\infty(I;L^2(\Omega))} + h\|\nabla u\|_{L^\infty(I;L^2(\Omega))}\right) \left(\|P_h\tilde{\delta}\|_{L^1(\Omega)} + \|\sigma^{\frac{N}{2}} P_h\tilde{\delta}\|_{L^2(\Omega)}\right) \\ &\le Cd^{-\frac{N}{2}} \ln\frac{T}{k}|\ln h| \left(\|u\|_{L^\infty(I;L^2(\Omega))} + h\|\nabla u\|_{L^\infty(I;L^2(\Omega))}\right), \end{aligned}$$

where in the last step we again used the stability of the L^2 projection with respect to the L^1 norm, the fact that $\|\tilde{\delta}\|_{L^1(\Omega)} \le C$, and Lemma 3.2 for the term $\|\sigma^{\frac{N}{2}} P_h\tilde{\delta}\|_{L^2(\Omega)}$. Inserting this inequality into (7.3), we obtain

$$|u_{kh}(T,x_0)| \le C\ln\frac{T}{k}|\ln h| \left(\|u\|_{L^\infty(I\times B_{2d})} + d^{-\frac{N}{2}} \left(\|u\|_{L^\infty(I;L^2(\Omega))} + h\|\nabla u\|_{L^\infty(I;L^2(\Omega))}\right)\right).$$

Using that the cG(r)dG(q) method is invariant on $X^{q,r}_{k,h}$, by replacing u and u_{kh} with $u-\chi$ and $u_{kh}-\chi$ for any $\chi \in X^{q,r}_{k,h}$, we obtain Theorem 2.2 for the case $\tilde{t} = T$.

In the case $t_{M-1} < \tilde{t} < T$ we proceed as in the proof of Theorem 2.1 using the dual problem (6.7) instead of (6.1). Then, we proceed as in the above proof using in the last step the discrete maximal parabolic regularity from Lemma 5.3 instead of Lemma 5.2. This completes the proof.

REFERENCES

[1] N. Y. Bakaev, M. Crouzeix, and V. Thomée, *Maximum-norm resolvent estimates for elliptic finite element operators on nonquasiuniform triangulations*, M2AN Math. Model. Numer. Anal., 40 (2006), pp. 923–937 (2007).

[2] N. Y. Bakaev, V. Thomée, and L. B. Wahlbin, *Maximum-norm estimates for resolvents of elliptic finite element operators*, Math. Comp., 72 (2003), pp. 1597–1610 (electronic).

[3] J. H. Bramble, A. H. Schatz, V. Thomée, and L. B. Wahlbin, *Some convergence estimates for semidiscrete Galerkin type approximations for parabolic equations*, SIAM J. Numer. Anal., 14 (1977), pp. 218–241.

[4] H. Chen, *An L^2-and L^∞ Error Analysis for Parabolic Finite Element Equations with Applications to Superconvergence and Error Expansions*, PhD thesis, Universität Heidelberg, Germany, 1993.

[5] M. Dobrowolski, *L^∞-convergence of linear finite element approximation to nonlinear parabolic problems*, SIAM J. Numer. Anal., 17 (1980), pp. 663–674.

[6] M. Dobrowolski, *Über die Zeitabhängigkeit des Diskretisierungsfehlers bei parabolischen Anfangs-, Randwertproblemen*, Z. Angew. Math. Mech., 60 (1980), pp. T283–T284.

[7] J. Douglas, Jr., T. Dupont, and L. Wahlbin, *The stability in L^q of the L^2-projection into finite element function spaces*, Numer. Math., 23 (1974/75), pp. 193–197.

[8] K. Eriksson and C. Johnson, *Adaptive finite element methods for parabolic problems. I. A linear model problem*, SIAM J. Numer. Anal., 28 (1991), pp. 43–77.

[9] ———, *Adaptive finite element methods for parabolic problems. II. Optimal error estimates in $L_\infty L_2$ and $L_\infty L_\infty$*, SIAM J. Numer. Anal., 32 (1995), pp. 706–740.
[10] K. ERIKSSON, C. JOHNSON, AND S. LARSSON, *Adaptive finite element methods for parabolic problems. VI. Analytic semigroups*, SIAM J. Numer. Anal., 35 (1998), pp. 1315–1325 (electronic).
[11] K. ERIKSSON, C. JOHNSON, AND V. THOMÉE, *Time discretization of parabolic problems by the discontinuous Galerkin method*, RAIRO Modél. Math. Anal. Numér., 19 (1985), pp. 611–643.
[12] P. JAMET, *Galerkin-type approximations which are discontinuous in time for parabolic equations in a variable domain*, SIAM J. Numer. Anal., 15 (1978), pp. 912–928.
[13] P. LASAINT AND P.-A. RAVIART, *On a finite element method for solving the neutron transport equation*, in Mathematical aspects of finite elements in partial differential equations (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1974), Math. Res. Center, Univ. of Wisconsin-Madison, Academic Press, New York, 1974, pp. 89–123. Publication No. 33.
[14] D. LEYKEKHMAN, *Pointwise localized error estimates for parabolic finite element equations*, Numer. Math., 96 (2004), pp. 583–600.
[15] D. LEYKEKHMAN AND B. VEXLER, *Optimal a priori error estimates of parabolic optimal control problems with pointwise control*, SIAM J. Numer. Anal., 51 (2013), pp. 2797–2821.
[16] ———, *Discrete maximal parabolic regularity for Galerkin finite element methods*, submitted, Preprint arXiv:1505.04808v2, (2016).
[17] ———, *Finite element pointwise results on convex polyhedral domains*, accepted to SIAM J. Numer. Anal., (2016).
[18] ———, *A priori error estimates for three dimensional parabolic optimal control problems with pointwise control*, submitted, (2016).
[19] D. LEYKEKHMAN AND L. B. WAHLBIN, *A posteriori error estimates by recovered gradients in parabolic finite element equations*, BIT, 48 (2008), pp. 585–605.
[20] B. LI, *Maximum-norm stability and maximal L^p regularity of FEMs for parabolic equations with Lipschitz continuous coefficients*, Numerische Mathematik, (2015), pp. 1–28.
[21] B. LI AND W. SUN, *Maximal L^p analysis of finite element solutions for parabolic equations with nonsmooth coefficients in convex polyhedra*, Math. Comp., (2015). accepted.
[22] D. MEIDNER, R. RANNACHER, AND B. VEXLER, *A priori error estimates for finite element discretizations of parabolic optimization problems with pointwise state constraints in time*, SIAM J. Control Optim., 49 (2011), pp. 1961–1997.
[23] J. A. NITSCHE, *L_∞-convergence of finite element Galerkin approximations for parabolic problems*, RAIRO Anal. Numér., 13 (1979), pp. 31–54.
[24] J. A. NITSCHE AND M. F. WHEELER, *L_∞-boundedness of the finite element Galerkin operator for parabolic problems*, Numer. Funct. Anal. Optim., 4 (1981/82), pp. 325–353.
[25] R. H. NOCHETTO AND C. VERDI, *Convergence past singularities for a fully discrete approximation of curvature-driven interfaces*, SIAM J. Numer. Anal., 34 (1997), pp. 490–512.
[26] C. PALENCIA, *Maximum norm analysis of completely discrete finite element methods for parabolic problems*, SIAM J. Numer. Anal., 33 (1996), pp. 1654–1668.
[27] A. PAZY, *Semigroups of linear operators and applications to partial differential equations*, vol. 44 of Applied Mathematical Sciences, Springer-Verlag, New York, 1983.
[28] R. RANNACHER, *L^∞-stability estimates and asymptotic error expansion for parabolic finite element equations*, in Extrapolation and defect correction (1990), vol. 228 of Bonner Math. Schriften, Univ. Bonn, Bonn, 1991, pp. 74–94.
[29] T. RICHTER, A. SPRINGER, AND B. VEXLER, *Efficient numerical realization of discontinuous Galerkin methods for temporal discretization of parabolic problems*, Numer. Math., 124 (2013), pp. 151–182.
[30] P. H. SAMMON, *Convergence estimates for semidiscrete parabolic equation approximations*, SIAM J. Numer. Anal., 19 (1982), pp. 68–92.
[31] A. H. SCHATZ, *A weak discrete maximum principle and stability of the finite element method in L_∞ on plane polygonal domains. I*, Math. Comp., 34 (1980), pp. 77–91.
[32] A. H. SCHATZ, V. THOMÉE, AND L. B. WAHLBIN, *Stability, analyticity, and almost best approximation in maximum norm for parabolic finite element equations*, Comm. Pure Appl. Math., 51 (1998), pp. 1349–1385.
[33] A. H. SCHATZ, V. C. THOMÉE, AND L. B. WAHLBIN, *Maximum norm stability and error estimates in parabolic finite element equations*, Comm. Pure Appl. Math., 33 (1980), pp. 265–304.
[34] A. H. SCHATZ AND L. B. WAHLBIN, *Interior maximum norm estimates for finite element methods*, Math. Comp., 31 (1977), pp. 414–442.
[35] ———, *On the quasi-optimality in L_∞ of the $\dot{H}^1$-projection into finite element spaces*, Math. Comp., 38 (1982), pp. 1–22.
[36] ———, *Interior maximum-norm estimates for finite element methods. II*, Math. Comp., 64 (1995), pp. 907–928.
[37] M. SCHMICH AND B. VEXLER, *Adaptivity with dynamic meshes for space-time finite element discretizations of parabolic equations*, SIAM J. Sci. Comput., 30 (2007/08), pp. 369–393.
[38] D. SCHÖTZAU AND T. P. WIHLER, *A posteriori error estimation for hp-version time-stepping methods for parabolic partial differential equations*, Numer. Math., 115 (2010), pp. 475–509.
[39] Z. W. SHEN, *Resolvent estimates in L^p for elliptic systems in Lipschitz domains*, J. Funct. Anal., 133 (1995), pp. 224–251.
[40] V. THOMÉE, *Galerkin finite element methods for parabolic problems*, vol. 25 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, second ed., 2006.
[41] V. THOMÉE AND L. B. WAHLBIN, *Stability and analyticity in maximum-norm for simplicial Lagrange finite element semidiscretizations of parabolic equations with Dirichlet boundary conditions*, Numer. Math., 87 (2000), pp. 373–389.
[42] F. TRÖLTZSCH, *Optimal control of partial differential equations*, vol. 112 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2010. Theory, methods and applications, Translated from the 2005 German original by Jürgen Sprekels.
[43] L. B. WAHLBIN, *A quasioptimal estimate in piecewise polynomial Galerkin approximation of parabolic problems*, in Numerical analysis (Dundee, 1981), vol. 912 of Lecture Notes in Math., Springer, Berlin-New York, 1982, pp. 230–245.
[44] L. B. WAHLBIN, *Local behavior in finite element methods*, in Handbook of numerical analysis, Vol. II, Handb. Numer. Anal., II, North-Holland, Amsterdam, 1991, pp. 353–522.